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Effective medium method in the problem of axial elastic shear wave propagation through fiber composites

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Abstract

The effective medium method (EMM) is applied to the solution of the problem of monochromatic elastic shear wave propagation through matrix composite materials reinforced with cylindrical unidirected fibers. The dispersion equations for the wave numbers of the mean wave field in such composites are derived using two different versions of the EMM. Asymptotic solutions of these equations in the long and short wave regions are found in closed analytical forms. Numerical solutions of the dispersion equations are constructed in a wide region of frequencies of the incident field that covers long, middle and short wave regions of the mean wave field. Velocities and attenuation factors of the mean wave fields in the composites obtained by different versions of the EMM are compared for various volume concentrations and properties of the inclusions. The main discrepancies in the predictions of different versions of the EMM are indicated, analyzed and discussed.

Keywords: Fiber composites; Wave propagation; Phase velocity; Attenuation factor; Self-consistent methods

1. Introduction

The problem of the monochromatic wave propagation through composite materials has many important applications. The solution of this problem allows us to predict the response of composite materials to various types of dynamic loading; this problem is a theoretical background of the non-destructive analysis of microstructures of composites by using ultrasonic technique. The main objectives of the theory in this problem are the dependence of the phase velocity and attenuation factor of the mean wave field propagating through the composite on the frequency of the incident field (dispersion curves) and on the details of the composite microstructure. For the composite materials with random microstructures this problem can not be solved exactly and only approximate solutions are available. The effective medium method (EMM) is widely used for the construction of such approximate solutions of the elastic wave propagation problem

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(see e.g., Sabina and Willis, 1988, 1993a,b; Jin-Yeon et al., 1995; Kanaun, 1996, 1997 and others). In the case of composites with spherical inclusions the predictions of the EMM are in agreement with known experimental data (see Sabina and Willis, 1988; Jin-Yeon et al., 1995).

In the literature there exist several different versions of the EMM, and one can say that the EMM is a group of self-consistent methods joined by a common hypothesis that for the construction of the wave field inside a typical inclusion in the composite the inhomogeneous material outside some vicinity of this inclusion may be changed for the homogeneous medium with effective (overall) properties of the composite. The analysis of various versions of the EMM in the case of electromagnetic wave propagation through heterogeneous media is presented in Kanaun (2000).

For the application of the EMM to the solution of the problem of elastic wave propagation through composite materials one has to understand the differences in predictions of various versions of the method, the character of possible errors and the area of application of every version. In this work we consider the propagation of monochromatic elastic shear waves in the composites with infinite unidirected cylindrical fibers when the wave vector of the incident field is orthogonal to the fiber directions and the polarization vector coincides with this direction (the axial shear waves). For this type of the waves we develop the version of the EMM that is similar to the one proposed by Budiansky (1965) and Hill (1965a,b) for the calculation of static properties of composite materials (version I). In this version every inclusion in the composite is considered as an isolated one embedded in the homogeneous medium with the effective properties of the composite. Another version of the EMM that corresponds better to experimental data was proposed in the works of Kerner (1956) and Christensen and Lo (1979). In this version a layer of matrix material was involved in the border between the effective medium and the inclusion (version II).

In the work these two versions of the EMM are generalized for the dynamic case. In Section 3, we develop a general dispersion equation of both versions of the EMM that serves for all frequencies of the incident field. The solutions of the dispersion equations give us the velocities and attenuation factors of the mean wave field propagating through the composite medium. Every version of the EMM reduce the problem of interactions between many inclusions in the composite to a specific one particle problem. Exact solutions of the one particle problems of two versions of the EMM are presented in Sections 4 and 6. In Section 4.2, we construct an approximate solution of the one particle problem that serves only in the long wave region. A similar approximation was used for the solution of the problem of long elastic wave propagation through particulate composite materials in Sabina and Willis (1988, 1993a,b).

Note that in the literature the EMM was used for the calculation of the wave velocities and attenuation factors of the mean wave field in the long and middle wave regions, where the wavelength of the mean wave field is more than a typical size of inclusions or is compared with the latter. In this work we consider the solutions of the dispersion equations of the EMM in a wide region of frequencies that covers long, middle and short wave regions. In the long and short wave regions we find asymptotic solutions of the dispersion equations in closed analytical forms (Sections 5 and 6). Numerical solutions of the dispersion equations of both versions of the EMM are obtained and compared in Section 7. The cases of the composites with contrast properties of components are considered. The discussion of the discrepancies in the predictions of various versions of the EMM in a wide region of frequencies of the incident field and volume concentrations of inclusions is presented in Section 8.

2. Integral equations of the diffraction problem

Let us consider an infinite homogeneous medium (matrix) with elastic moduli tensor C^0 and density ρ_0 containing a random set of continuous cylindrical fibers directed along x_3-axis. C and ρ are the elastic moduli tensor and density of the fibers. If a monochromatic axial shear wave of frequency ω propagates in such a composite, and the dependence on time is defined by the multiplier $e^{i\omega t}$, only the component u_3 of the amplitude of the displacement field $u_i(x)$ is not equal to zero ("antiplane" strain state)

$$u_1 = u_2 = 0, \qquad u_3(x) = u(y), \qquad x = x(x_1, x_2, x_3), \qquad y = y(x_1, x_2) \tag{2.1}$$

and the equation of motion takes the form

$$\partial_i C_{i3j3}(y)\partial_{jk} u(y) + \rho(y)\omega^2 u(y) = 0, \qquad \partial_i = \frac{\partial}{\partial x_i}. \tag{2.2}$$

The components $C_{i3j3}(y) = \mu(y)\delta_{ij}$ of the elastic moduli tensor $C_{ijkl}(y)$ of the medium and its density $\rho(y)$ may be presented as the following sums:

$$\mu(y) = \mu_0 + \mu_1 S(y), \qquad \rho(y) = \rho_0 + \rho_1 S(y), \qquad \mu_1 = \mu - \mu_0, \qquad \rho_1 = \rho - \rho_0. \tag{2.3}$$

Here $\mu(y)$ is the elastic shear modulus in the direction of the fibers, $S(y)$ is the characteristic function of the region S occupied by the inclusions ($S(y) = 1$ if $y \in S$, $S(y) = 0$ if $y \notin S$). After substituting Eq. (2.3) into Eq. (2.2) the latter takes the following form:

$$\mu_0 \Delta u(y) + \rho_0 \omega^2 u(y) = -\mu_1 \partial_i[\varepsilon_i(y)S(y)] - \omega^2 \rho_1 u(y) S(y), \tag{2.4}$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the Laplace operator, $\varepsilon_i(y) = \partial_i u(y)$. Applying the operator $(\mu_0\Delta + \rho_0\omega^2)^{-1}$ to both sides of this equation we obtain the integral equation for the displacement field $u(y)$ in the form

$$u(y) = u_0(y) + \int \partial_i G(y - y')\mu_1 \varepsilon_i(y')S(y')\,dy' + \omega^2 \int G(y - y')\rho_1 u(y')S(y')\,dy'. \tag{2.5}$$

Here $u_0(y)$ is the incident field that would have existed in the medium without inclusions ($\mu_1 = 0$, $\rho_1 = 0$). $G(y)$ is the Green function of the operator $\mu_0\Delta + \rho_0\omega^2$. The equation for this function and its solution have the forms (see Eringen and Suhubi, 1975)

$$(\mu_0\Delta + \rho_0\omega^2)G(y) = -\delta(y),$$

$$G(y) = -\frac{i}{4\mu_0} H_0(k_0 r), \quad r = |\mathbf{y}|, \;\; k_0 = \omega\sqrt{\frac{\rho_0}{\mu_0}}, \tag{2.6}$$

where $\delta(y)$ is Dirac's delta-function, $H_0(z)$ is the Hankel function of the second kind and zero-order. Integration in Eq. (2.5) is spread over entire 2D-space. Note that the Fourier transform of the function $G(y)$ has the form

$$G(\mathbf{k}) = L_0^{-1}(\mathbf{k}), \qquad L_0(\mathbf{k}) = \mu_0 k^2 - \rho_0\omega^2, \quad k = |\mathbf{k}|, \tag{2.7}$$

where $\mathbf{k}(k_1, k_2)$ is the vector parameter of the Fourier transform. (We denote the Fourier transforms of functions by the same letter with the other argument only.)

The incident field $u_0(y)$ in Eq. (2.5) is a plane shear wave with the wave vector $\mathbf{k}^0 = k_0\mathbf{n}^0$ that is orthogonal to x_3-axis. For such a field we have

$$u_0(y) = U_0 e^{-i\mathbf{k}^0\cdot\mathbf{y}}, \qquad \varepsilon_i^0(y) = \partial_i u_0(y) = -ik_0 n_i^0 U_0 e^{-i\mathbf{k}^0\cdot\mathbf{y}}. \tag{2.8}$$

The equation for the strain field $\varepsilon_i(y)$ follows from Eq. (2.5) in the form

$$\varepsilon_i(y) = \varepsilon_i^0(y) + \int [\partial_i\partial_k G(y - y')\mu_1\varepsilon_k(y') + \omega^2\partial_i G(y - y')\rho_1 u(y')]S(y')\,dy'. \tag{2.9}$$

Note that the multiplier $S(y)$ in the right-hand sides of Eqs. (2.5) and (2.9) cuts the functions $u(y)$ and $\varepsilon(y)$ in the region occupied by the inclusions. Thus, the main unknowns of the problem are the fields inside inclusions. If these fields are known, the wave fields in the matrix may be reconstructed from Eqs. (2.5) and (2.9).

3. General scheme of the basic version of the EMM

If the incident field is a plane monochromatic wave, the field propagating in the composite with a random set of cylindrical fibers will be a non-plane random field. Our objective is to evaluate the mean value of this field. Strictly speaking, in order to construct the mean wave field one has to solve the wave problem for every realization of the random set of inclusions and then to average the obtained solutions over the ensemble of realizations of this set. The difficulties of this problem oblige us to find its approximate solution. In this work we use the effective medium method for the construction of such a solution.

Let us consider a realization of a homogeneous random set of inclusions in the matrix. In order to find the mean wave field in the composite we accept the following two hypotheses:

1. *Every inclusion in composite behaves as isolated one embedded in the homogeneous medium with the effective properties of the composite. The field that acts on this inclusion is a plane wave propagating in the effective medium.*
2. *The mean wave field in the composite coincides with the field propagating in the homogeneous effective medium.*

These two hypotheses correspond to the version of the EMM proposed by Budiansky (1965) and Hill (1965a,b) for the calculation of static elastic moduli of composite materials.

The first hypothesis reduces the problem of interactions between many inclusions in the composite to a one particle problem. This problem is diffraction of a plane shear monochromatic wave on an isolated fiber embedded in the effective homogeneous medium with the properties μ_*, ρ_*. The integral equations of this problem are similar to Eqs. (2.5) and (2.9) and have the forms

$$u(y) = u^*(y) + \int_{s_0} [\partial_i G_*(y-y')\mu_{*1}\varepsilon_i(y') + \omega^2 G_*(y-y')\rho_{*1}u(y')]\,\mathrm{d}y', \tag{3.1}$$

$$\varepsilon_i(y) = \varepsilon_i^*(y) + \int_{s_0} [\partial_i\partial_j G_*(y-y')\mu_{*1}\varepsilon_j(y') + \omega^2 \partial_i G_*(y-y')\rho_{*1}u(y')]\,\mathrm{d}y'. \tag{3.2}$$

Here s_0 is the area of the fiber cross-section, $G_*(y)$ is the Green's function of the homogeneous medium with the effective properties μ_* and ρ_* of the composite, $\mu_{*1} = \mu - \mu_*$, $\rho_{*1} = \rho - \rho_*$. The displacement $u^*(y)$ and strain $\varepsilon_i^*(y)$ are plane waves with the wave vector $\mathbf{k}^*$ propagating in the effective medium

$$u_*(u) = U_* e^{-i\mathbf{k}^*\cdot\mathbf{y}}, \qquad \varepsilon_j^*(y) = -ik_j^* U_* e^{-i\mathbf{k}^*\cdot\mathbf{y}}, \qquad \mathbf{k}^* = k_*\mathbf{n}, \qquad k_* = \omega\sqrt{\frac{\rho_*}{\mu_*}}. \tag{3.3}$$

If the distribution of fibers in the matrix is homogeneous and isotropic, the effective medium is transversely isotropic, and the effective wave vector $\mathbf{k}^*$ and the wave vector $\mathbf{k}^0$ of the incident field have the same direction.

Let the general solution of Eqs. (3.1) and (3.2) be known, and the fields $u(y)$ and $\varepsilon_i(y)$ inside the inclusion with the center at point $y^0 = 0$ be presented in the form

$$u(y) = (\Lambda u_*)(y) = \Lambda[U_* e^{-i\mathbf{k}^*\cdot\mathbf{y}}], \qquad \varepsilon_i(y) = \partial_i(\Lambda u_*)(y). \tag{3.4}$$

Here Λ is a linear operator that depends on the dynamic properties of the effective medium and inclusion.

If the inclusion occupies area S_0 with the center at a point $y^0 \neq 0$ one can present the field $u(y)$ inside such an inclusion in the form ($y \in S_0$)

$$\begin{aligned} u(y) &= \Lambda[U_* \mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot(\mathbf{y}-\mathbf{y}^0)}\mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot\mathbf{y}^0}] = \Lambda[\mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot(\mathbf{y}-\mathbf{y}^0)}]U_*\mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot\mathbf{y}^0} = \Lambda[\mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot(\mathbf{y}-\mathbf{y}^0)}]\mathrm{e}^{\mathrm{i}\mathbf{k}^*\cdot(\mathbf{y}-\mathbf{y}^0)}U_*\mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot\mathbf{y}} \\ &= \Lambda^u(y-y^0)u_*(y), \quad \Lambda^u(z) = \Lambda[\mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot\mathbf{z}}]\mathrm{e}^{\mathrm{i}\mathbf{k}^*\cdot\mathbf{z}}. \end{aligned} \tag{3.5}$$

In the same way for the field $\varepsilon_i(y) = \partial_i u(y)$ we have

$$\varepsilon_i(y) = \partial_i \Lambda[U_*\mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot(\mathbf{y}-\mathbf{y}^0)}\mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot\mathbf{y}^0}] = \Lambda_i^\varepsilon(y-y^0)u_*(y),$$

$$\Lambda_i^\varepsilon(z) = (\partial_i\Lambda[\mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot\mathbf{z}}])\mathrm{e}^{\mathrm{i}\mathbf{k}^*\cdot\mathbf{z}}. \tag{3.6}$$

Here we take into account linearity of the operator Λ. Note that the functions $\Lambda^u(z)$ and $\Lambda_i^\varepsilon(z)$ do not depend on the position y^0 of the center of the inclusion. These functions may be constructed from the solution of the one particle problem for the inclusion centered at point $y = 0$.

Let us introduce stationary random functions $\lambda^u(y)$ and $\lambda^\varepsilon(y)$ in 2D-space. These functions coincide with $\Lambda^u(y-y^i)$ and $\Lambda^\varepsilon(y-y^i)$ if y is inside the inclusion centered at point y^i ($i = 1, 2, 3, \ldots$), and they are equal to zero in the matrix. Using these functions and hypothesis 1 of the EMM one can present the wave field $u(y)$ in the composite in the form that follows from Eqs. (2.5), (3.5) and (3.6)

$$u(y) = u_0(y) + \int \left[\partial_i G(y-y')\mu_1\lambda_i^\varepsilon(y')u_*(y') + \omega^2 G(y-y')\rho_1\lambda^u(y')u_*(y')\right]S(y')\,\mathrm{d}y'. \tag{3.7}$$

Here $u_*(y) = U_*\mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot\mathbf{y}}$ is a plane wave with the effective wave vector $\mathbf{k}^*$.

In order to find the mean value of the random wave field $u(y)$ let us average both parts of Eq. (3.7) over ensemble realizations of the random set of inclusions and take into account the condition of self-consistency (hypothesis 2)

$$u_*(y) = \langle u(y)\rangle. \tag{3.8}$$

As a result we obtain the integral equation for the mean wave field $\langle u(y)\rangle$ in the form

$$\langle u(y)\rangle = u_0(y) + p\int [\partial_i G(y-y')\mu_1\Lambda_i^C + \omega^2 G(y-y')\rho_1\Lambda_\rho]\langle u(y')\rangle\,\mathrm{d}y', \tag{3.9}$$

$$\Lambda_\rho(\mathbf{k}^*) = \frac{1}{p\Omega}\lim_{\Omega\to\infty}\int_\Omega \lambda^u(y)\,\mathrm{d}y = \frac{1}{\langle s\rangle}\left\langle \int_s \Lambda^u(y)\,\mathrm{d}y\right\rangle, \tag{3.10}$$

$$\Lambda_i^C(\mathbf{k}^*) = \frac{1}{p\Omega}\lim_{\Omega\to\infty}\int_\Omega \lambda_i^\varepsilon(y)\,\mathrm{d}y = \frac{1}{\langle s\rangle}\left\langle \int_s \Lambda_i^\varepsilon(y)\,\mathrm{d}y\right\rangle. \tag{3.11}$$

Here Λ_ρ and Λ_i^C are constant (with respect to spatial coordinates) scalar and vector, Ω is an arbitrary region that occupies entire 2D-plane (x_1, x_2) in the limit $\Omega \to \infty$, p is the volume concentration of inclusions, s is the area occupied by a typical inclusion. The averaging in Eqs. (3.10) and (3.11) is over the ensemble of realizations of random sizes of inclusions.

Let us apply the Fourier transform to Eq. (3.9) and multiply the result with $L_0(\mathbf{k}) = \mu_0 k^2 - \rho_0\omega^2$. Taking into account the equations

$$L_0(\mathbf{k})G(\mathbf{k}) = 1, \qquad L_0(\mathbf{k})u_0(\mathbf{k}) = 0 \tag{3.12}$$

that follow from Eqs. (2.7) and (2.8), we obtain the equation for $\langle u(\mathbf{k})\rangle$ in the form

$$L_*(\mathbf{k})\langle u(\mathbf{k})\rangle = 0, \qquad L_*(\mathbf{k}) = L_0(\mathbf{k}) + p\mu_1 \mathrm{i}k_i\Lambda_i^C(\mathbf{k}^*) - p\rho_1\omega^2\Lambda_\rho(\mathbf{k}^*). \tag{3.13}$$

Because vector Λ_i^C in Eq. (3.11) is a function of the vector $\mathbf{k}^*$ only, Λ_i^C may be written in the form

$$\Lambda_j^C(\mathbf{k}^*) = -\mathrm{i}k_j^* H_C(k^*), \quad k^* = |\mathbf{k}^*|, \tag{3.14}$$

where $H_C(k^*)$ is a scalar function. If the mean wave field $\langle u(y)\rangle$ is a plane wave ($\langle u(y)\rangle = U_* \mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot\mathbf{y}}$), its Fourier transform is $\langle u(\mathbf{k})\rangle = (2\pi)^2 U_* \delta(\mathbf{k}-\mathbf{k}^*)$, and Eq. (3.13) takes the form $L_*(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}^*) = 0$ or

$$L_*(\mathbf{k}^*) = L_0(\mathbf{k}^*) + p\mu_1 (k^*)^2 H_C(k^*) - p\rho_1\omega^2 \Lambda_\rho(k^*) = 0. \tag{3.15}$$

This equation may be also written in the form

$$\mu_*(k_*)k_*^2 - \rho_*(k_*)\omega^2 = 0, \tag{3.16}$$

$$\mu_*(k_*) = \mu_0 + p\mu_1 H_C(k_*), \qquad \rho_*(k_*) = \rho_0 + p\rho_1 \Lambda_\rho(k^*). \tag{3.17}$$

Eq. (3.16) is the dispersion equation for the effective wave number k_* of the mean wave field in the composite. The functions H_C and Λ_ρ are to be found from the solution of the one particle problem (3.1), (3.2). Thus, Eqs. (3.16) and (3.17) are the system for the effective parameters μ_*, ρ_*, k_* of the composite in the framework of the EMM. The phase velocity v_* and attenuation factor γ of the mean wave field $\langle u(y)\rangle$ are connected with the wave number k_* by the equations

$$v_* = \frac{\omega}{\mathrm{Re}(k_*)}, \qquad \gamma = -\mathrm{Im}(k_*). \tag{3.18}$$

4. The solution of the one particle problem

4.1. The exact solution

The one particle problem of the EMM is the solution of the integral equations (3.1) and (3.2) if s_0 is a disk of radius a centered at point $y = 0$. The integral equation (3.1) for $u(y)$ is equivalent to the following system of differential equations:

$$\begin{aligned} & u(y) = u^+(y), \qquad |y| < a, \qquad u(y) = u^-(y), \qquad |y| > a \\ & \Delta u^+ + k^2 u^+ = 0, \qquad |y| < a, \qquad k^2 = \frac{\rho\omega^2}{\mu}, \\ & \Delta u^- + k_*^2 u^- = 0, \qquad |y| > a, \qquad k_*^2 = \frac{\rho_*\omega^2}{\mu_*} \end{aligned} \tag{4.1}$$

with the conditions on the boundary of s_0 ($r = a$)

$$u^+(a,\varphi) = u^-(a,\varphi), \qquad \mu \left.\frac{\partial u^+(r,\varphi)}{\partial r}\right|_{r=a} = \mu_* \left.\frac{\partial u^-(r,\varphi)}{\partial r}\right|_{r=a}. \tag{4.2}$$

Here r and φ are the polar coordinates in the y-plane with the origin at the center of the inclusion. The field $u^-(y)$ should satisfy the radiation condition at infinity

$$u^-(y) - u_*(y) \sim \frac{\exp(-\mathrm{i}k_* r)}{\sqrt{r}}, \quad r = |y| \to \infty, \tag{4.3}$$

where $u_*(y)$ is the incident field.

The solution of this problem has the form (see Eringen and Suhubi, 1975)

$$u^{+}(y) = U_{*}\sum_{m=0}^{\infty} a_m J_m(kr)\cos(m\varphi), \tag{4.4}$$

$$u^{-}(y) = u_{*}(y) + U_{*}\sum_{m=0}^{\infty} b_m H_m(k_{*}r)\cos(m\varphi), \tag{4.5}$$

$$u_{*}(y) = U_{*}\exp(-\mathrm{i}\mathbf{k}^{*}\cdot\mathbf{y}) = U_{*}\sum_{m=0}^{\infty}[\epsilon_m(-\mathrm{i})^m J_m(k_{*}r)],$$

where $\epsilon_m = 1$ if $m = 0$, $\epsilon_m = 2$ if $m > 0$, $J_m(z)$ are Bessel functions and $H_m(z)$ are Hankel functions of the second kind. The coefficients a_m in Eq. (4.4) take the forms

$$a_m = (-\mathrm{i})^m \epsilon_m \frac{\mu_{*}k_{*}}{\Delta_m}, \tag{4.6}$$

$$\Delta_m = \frac{\mathrm{i}\pi}{2}k_{*}a[\mu_{*}k_{*}H'_m(k_{*}a)J_m(ka) - \mu k H_m(k_{*}a)J'_m(ka)], \qquad f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z}. \tag{4.7}$$

After substituting Eq. (4.4) into Eqs. (3.10), (3.11), and (3.14) for Λ_ρ and H_C we obtain

$$\Lambda_\rho = \sum_{m=0}^{\infty} a_m g_m, \qquad H_C = \sum_{m=0}^{\infty} a_m g_{1m}, \tag{4.8}$$

$$g_m = \frac{2i^m}{a}\cdot\frac{1}{k^2 - k_{*}^2}[kJ_{m+1}(ka)J_m(k_{*}a) - k_{*}J_m(ka)J_{m+1}(k_{*}a)], \tag{4.9}$$

$$g_{1m} = g_m + \frac{2i^m}{ak_{*}}J_m(ka)J'_m(k_{*}a), \qquad J'_m(z) = \frac{\mathrm{d}J_m(z)}{\mathrm{d}z}. \tag{4.10}$$

The wave field (4.5) outside the inclusion consists of two parts: the incident field $u_{*}(y)$ and the field $u^{s}(y)$ scattered on the inclusion. Let us consider the diffraction of a plane wave propagating in the original matrix ($\mathbf{k}^{*} = \mathbf{k}^{0}, u_0(y) = U_0\exp(-\mathrm{i}\mathbf{k}^{0}\cdot\mathbf{y})$) on an isolated cylindrical inclusion. In this case the field $u^{s}(y)$ takes the form

$$u^{s}(y) = u(y) - u_0(y) = U_0\sum_{m=0}^{\infty} b_m H_m(k_0 r)\cos(m\varphi), \quad |y| > a. \tag{4.11}$$

On the other hand, the field scattered on the inclusion is the integral terms in Eq. (3.1), where the effective medium should be replaced with the original matrix. Thus, $u_s(y)$ is also presented in the form

$$u^{s}(y) = \int_{s_0}[\partial_i G(y - y')\mu_1\varepsilon_i(y') + \omega^2 G(y - y')\rho_1 u(y')]\,\mathrm{d}y'. \tag{4.12}$$

Because integration here spreads over the region s_0 only, Eq. (4.12) defines the scattered field via the fields u^{+} and ε_i^{+} inside the inclusion.

Let us consider the long distance asymptotic of the scattered field. Using a standard technique of the evaluation of the integral in Eq. (4.12) (see Eringen and Suhubi, 1975; Bohren and Huffman, 1983) we obtain that for large $|y|$ the following equation holds:

$$u^s(y) \approx A(\mathbf{n})\frac{\mathrm{e}^{-\mathrm{i}k_0 r}}{\sqrt{r}}, \quad \mathbf{n} = \frac{\mathbf{y}}{r}, \quad r = |\mathbf{y}|, \tag{4.13}$$

$$A(\mathbf{n}) = \mathrm{i}\sqrt{\frac{k_0^3}{8\pi}}\mathrm{e}^{\mathrm{i}\pi/4}\left[\frac{\mathrm{i}\mu_1}{k_0\mu_0}n_k\int_{s_0}\varepsilon_k(y')\mathrm{e}^{\mathrm{i}k_0(\mathbf{n}\cdot\mathbf{y}')}\,\mathrm{d}y' - \frac{\rho_1}{\rho_0}\int_{s_0}u(y')\mathrm{e}^{\mathrm{i}k_0(\mathbf{n}\cdot\mathbf{y}')}\,\mathrm{d}y'\right], \tag{4.14}$$

where $A(\mathbf{n})$ is the far amplitude of the scattered field. If vector $\mathbf{n}$ has the direction of the wave vector $\mathbf{k}^0 = k_0\mathbf{n}^0$ of the incident field, $A(\mathbf{n}^0)$ is the forward scattering amplitude. The long distance asymptotics of the stress and strain scattered fields take the forms that follow from Eq. (4.13)

$$\varepsilon_i^s = -\mathrm{i}k_0 n_i A(\mathbf{n})\frac{\mathrm{e}^{\mathrm{i}k_0 r}}{\sqrt{r}}, \qquad \sigma_i^s = -\mathrm{i}k_0 n_i \mu_0 A(\mathbf{n})\frac{\mathrm{e}^{\mathrm{i}k_0 r}}{\sqrt{r}}. \tag{4.15}$$

The total scattering cross-section of the inclusion $Q(k_0)$ is defined by the equation (see Eringen and Suhubi, 1975)

$$Q(k_0) = \frac{\mathrm{Im}[J(k_0)]}{k_0\mu_0}, \qquad J(k_0) = \int_\Gamma(\sigma_i^0\widetilde{u}_s + \sigma_i^s\widetilde{u}_0)n_i\,\mathrm{d}\Gamma, \tag{4.16}$$

where $u_0(y)$ and $\sigma_k^0(y)$ are plane incident waves with the wave vector $\mathbf{k}^0 = k_0\mathbf{n}^0$

$$u_0 = \mathrm{e}^{-\mathrm{i}k_0\mathbf{n}^0\cdot\mathbf{y}}, \qquad \sigma_i^0 = -\mathrm{i}k_0 n_i^0\mu_0\mathrm{e}^{-\mathrm{i}k_0\mathbf{n}^0\cdot\mathbf{y}}. \tag{4.17}$$

Tildes in Eq. (4.16) mean the complex conjugations, Γ is a circle of a large radius R. It follows from Eqs. (4.13)–(4.15) that

$$(\sigma_i^0\widetilde{u}_s + \sigma_i^s\widetilde{u}_0)n_i = -\frac{\mathrm{i}k_0\mu_0}{\sqrt{r}}[A(\mathbf{n})\mathrm{e}^{-\mathrm{i}k_0 r}\mathrm{e}^{\mathrm{i}k_0\mathbf{n}^0\cdot\mathbf{y}} + (\mathbf{n}^0\cdot\mathbf{n})\widetilde{A}(\mathbf{n})\mathrm{e}^{\mathrm{i}k_0 r}\mathrm{e}^{-\mathrm{i}k_0\mathbf{n}^0\cdot\mathbf{y}}]. \tag{4.18}$$

After substituting this equation into Eq. (4.16) and using the saddle-point method for the evaluation of the integral $J(k_0)$ when $R \to \infty$, we go to the following expression for $J(k_0)$

$$J(k_0) = \mathrm{i}\mu_0\sqrt{2\pi k_0}\left[A(\mathbf{n}^0)\mathrm{e}^{-\mathrm{i}\pi/4} + \widetilde{A}(\mathbf{n}^0)\mathrm{e}^{\mathrm{i}\pi/4}\right] = -\mathrm{i}\mu_0\sqrt{8\pi k_0}\mathrm{Re}\left[A(\mathbf{n}^0)\mathrm{e}^{-\mathrm{i}\pi/4}\right]. \tag{4.19}$$

Finally, for the total scattering cross-section $Q(k_0)$ in Eq. (4.16) we obtain the equation

$$Q(k_0) = -\sqrt{\frac{8\pi}{k_0}}\mathrm{Re}\left[A(\mathbf{n}^0)\mathrm{e}^{-\mathrm{i}\pi/4}\right]. \tag{4.20}$$

Thus, the cross-section $Q(k_0)$ is defined via the forward scattering amplitude $A(\mathbf{n}^0)$. (This is the optical theorem for an infinite cylinder; see a similar theorem for electro-magnetic waves in Bohren and Huffman (1983).)

From Eqs. (3.5), (3.6), (3.10), (3.11) and (3.14) follows that the forward scattering amplitude $A(\mathbf{n}^0)$ in Eq. (4.14) takes the form

$$A(\mathbf{n}^0) = \mathrm{i}\pi a^2\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\frac{k_0^3}{8\pi}}\left(\frac{\mu_1}{\mu_0}H_C - \frac{\rho_1}{\rho_0}\Lambda_\rho\right), \tag{4.21}$$

where quantities H_C and Λ_ρ are

$$H_C = \frac{i}{\pi a^2 k_0}n_i^0\int_{S_0}\varepsilon_i(y)\mathrm{e}^{\mathrm{i}k_0(\mathbf{n}^0\cdot\mathbf{y})}\,\mathrm{d}y, \qquad \Lambda_\rho = \frac{1}{\pi a^2}\int_{S_0}u(y)\mathrm{e}^{\mathrm{i}k_0(\mathbf{n}^0\cdot\mathbf{y})}\,\mathrm{d}y. \tag{4.22}$$

After substituting here $u(y) = u^+(y), \varepsilon(y) = \partial_i u^+(y)$ from Eqs. (4.4) and (4.6), where $k_* = k_0$, $U_* = U_0 = 1$, we obtain for $Q(k_0)$ the following equation:

$$Q(k_0) = \pi a^2 k_0 \mathrm{Im}\left(\frac{\mu_1}{\mu_0} H_C - \frac{\rho_1}{\rho_0} \Lambda_\rho\right). \tag{4.23}$$

Here Λ_ρ and H_C have forms (4.8) if $k_* = k_0$.

Using the technique presented in Bohren and Huffman (1983) it is possible to show that the short wave limit of $Q(k_0)$ ($k_0 \to \infty$) takes the form (the paradox of extinction)

$$\lim_{k_0\to\infty} \frac{Q(k_0)}{\pi a} = \frac{4}{\pi}. \tag{4.24}$$

The character of convergence of $Q(k_0)$ to this limit may be seen from Fig. 1. $\overline{Q}(k_0)$ in this figure is

$$\overline{Q}(k_0) = k_0 a\left(\frac{\mu_1}{\mu_0} H_C - \frac{\rho_1}{\rho_0} \Lambda_\rho\right) = k_0 a \sum_{m=0}^{\infty} a_m \left(\frac{\mu_1}{\mu_0} g_{1m} - \frac{\rho_1}{\rho_0} g_m\right). \tag{4.25}$$

The case (a) corresponds to a heavy and hard inclusion ($\mu_0 = 1$, $\rho_0 = 1$, $\mu = 100$, $\rho_0 = 10$), the case (b) to a soft and light inclusion ($\mu_0 = 1$, $\rho_0 = 1$, $\mu = 0.01$, $\rho_0 = 0.1$). Thus, in the short wave limit ($k_0 \to \infty$) the following equations for the real and imaginary parts of $\overline{Q}(k_0)$ hold

$$\mathrm{Re}\,\overline{Q}(k_0) \to 0, \qquad \mathrm{Im}\,\overline{Q}(k_0) \to \frac{4}{\pi}. \tag{4.26}$$

This result does not depend on the properties of the matrix and inclusion.

4.2. An approximate solution of the one particle problem

In a number of publications (Sabina and Willis (1988, 1993a,b) and others) an approximate solution of the one particle problem (3.1) and (3.2) was used in the frame of the EMM. In this approximation elastic fields $u(y)$ and $\varepsilon_i(y)$ are assumed to be constant inside every inclusion. After substituting these constants

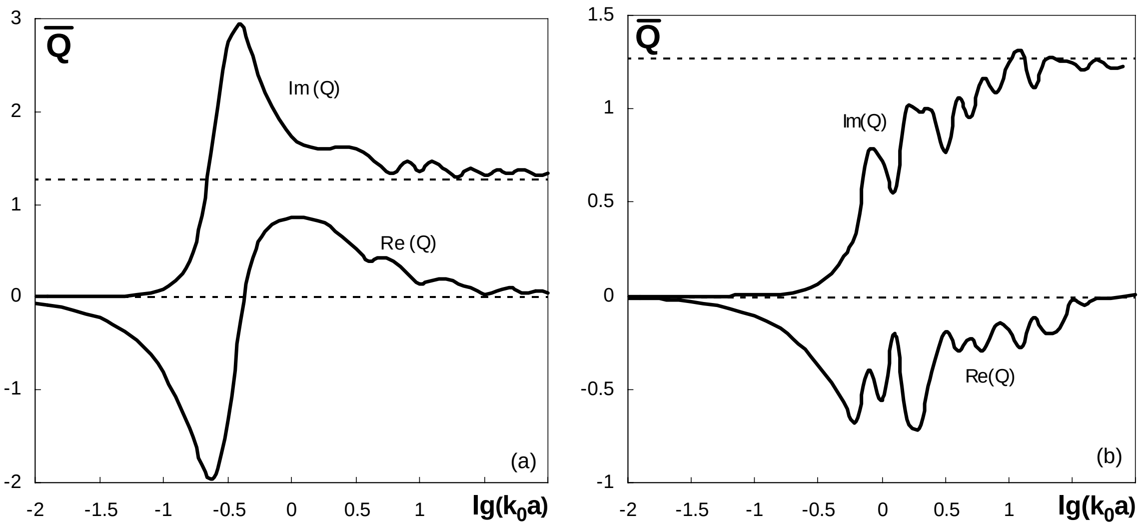

Fig. 1. The dependence of the normalized forward scattered amplitude $\overline{Q}$ of shear waves on an isolated fiber on the frequency of the incident field k_0: (a) a hard and heavy cylindrical inclusion ($\mu_0 = 1$, $\mu = 100$, $\rho_0 = 1$, $\rho = 10$); (b) a soft and light inclusion ($\mu_0 = 1$, $\mu = 0.01$, $\rho_0 = 1$, $\rho = 0.1$).

into the right-hand sides of Eqs. (3.1) and (3.2) and averaging the results over volume of the inclusion (Galerkin's scheme) we go to the following system of linear algebraic equations for u and ε_i

$$u = \overline{u}^* + \rho_{*1}\omega^2\overline{g}^* u, \qquad \varepsilon_i = \overline{\varepsilon}_i^* + \overline{P}_{ik}^*\mu_{*1}\varepsilon_k. \tag{4.27}$$

Here the overbear is the mean over the region s_0,

$$\overline{g}^*(y) = \frac{1}{s_0}\int_{s_0} \mathrm{d}y \int_{s_0} G_*(y-y')\,\mathrm{d}y', \qquad \overline{P}_{ik}^*(y) = \frac{1}{s_0}\int_{s_0} \mathrm{d}y \int_{s_0} \partial_i\partial_k G_*(y-y')\,\mathrm{d}y'. \tag{4.28}$$

The equation

$$\frac{1}{s_0}\int_{s_0} \mathrm{d}y \int_{s_0} \partial_i G_*(y-y')\,\mathrm{d}y' = 0 \tag{4.29}$$

that holds for a circular inclusion has been taken into account.

After calculating the integrals in Eqs. (4.27) and (4.28) we obtain

$$\overline{g}^*(y) = -\frac{1}{\rho_*\omega^2}\left[\frac{\mathrm{i}\pi}{2}J_0(k_*r)k_*aH_1^{(2)}(k_*a) + 1\right], \tag{4.30}$$

$$\overline{P}_{ik}^* = \frac{\mathrm{i}\pi}{2\mu_*}J_1(k_*a)H_1^{(2)}(k_*a)\theta_{ik}, \quad \theta_{ik} = \delta_{ik} - n_i n_k, \quad n_i = \frac{y_i}{r}, \tag{4.31}$$

$$\overline{u}^* = U_*\frac{1}{\pi a^2}\int_{S_0} \mathrm{e}^{-\mathrm{i}\mathbf{k}^*\cdot\mathbf{y}}\,\mathrm{d}y = U_* h(k_*a), \quad h(k_*a) = \frac{2J_1(k_*a)}{k_*a} \tag{4.32}$$

and the solution of Eq. (4.27) takes the form

$$u = (1 - \rho_{*1}\omega^2\overline{g}^*)^{-1}h(k_*a)U_*, \tag{4.33}$$

$$\varepsilon_i = \left[1 - \frac{\mu_{*1}}{\mu_*}\cdot\frac{\mathrm{i}\pi}{2}J_1(k_*a)H_1^{(2)}(k_*a)\right]^{-1} h(k_*a)\varepsilon_i^*, \quad \varepsilon_i^* = -\mathrm{i}k_i^* U_*. \tag{4.34}$$

Finally, from Eqs. (4.22) we obtain for Λ_ρ and H_C the following approximate equations

$$\Lambda_\rho = (1 - \rho_{*1}\omega^2\overline{g}^*)^{-1}h^2(k_*a), \tag{4.35}$$

$$H_C = \left[1 - \frac{\mu_{*1}}{\mu_*}\cdot\frac{\mathrm{i}\pi}{2}J_1(k_*a)H_1^{(2)}(k_*a)\right]^{-1} h^2(k_*a). \tag{4.36}$$

These equations serve only in the long wave region.

5. Solution of the dispersion equation in the long and short wave regions

In this section, we study asymptotic solutions of Eqs. (3.16) and (3.17) in the long and short wave regions. In the long wave region the wave number k_0 is small ($k_0a \ll 1$), and only the main terms in the real and imaginary parts of the coefficients a_m in Eq. (4.6) and coefficients g_m, g_{1m} in Eqs. (4.9) and (4.10) should be taken into account. Because the main terms of the real and imaginary parts of the Bessel and Hankel functions for small values of arguments are

$$J_n(z) \approx \frac{1}{n!}\left(\frac{z}{2}\right)^n, \qquad \frac{\mathrm{i}\pi}{2}z^n H_n(z) \approx -2^{n-1}(n-1)! + \mathrm{i}\pi\frac{z^{2n}}{n!2^{n+1}}, \tag{5.1}$$

the main terms of the coefficients a_m, g_m, g_{1m} take the forms

$$a_0 = \left[1 - \frac{i\pi}{4}(ka)^2\left(\frac{k_*^2}{k^2} - \frac{\mu}{\mu_*}\right)\right]^{-1}, \qquad a_1 = -\frac{2ik_*}{k}\left[1 + \frac{\mu_{*1}}{2\mu_*}\left(1 - \frac{i\pi}{4}(k_*a)^2\right)\right]^{-1},$$

$$g_0 = 1, \qquad g_{10} = 0, \qquad g_{11} = \frac{ik}{2k_*}. \tag{5.2}$$

The other coefficients a_m, g_m, g_{1m} may be neglected in the long wave region. As a result the main terms of the quantities Λ_ρ and H_C in Eq. (4.8) take the forms

$$\Lambda_\rho = 1 - \frac{i\pi}{4}(k_*a)^2\frac{\rho_{*1}}{\rho_*}, \tag{5.3}$$

$$H_C = \left(1 + \frac{\mu_{*1}}{2\mu_*}\right)^{-1}\left[1 + \frac{i\pi}{8}(k_*a)^2\frac{\mu_{*1}}{\mu_*}\left(1 + \frac{\mu_{*1}}{2\mu_*}\right)^{-1}\right]. \tag{5.4}$$

Note that the approximate solution of the one particle problem presented in Section 4.2 gives the same asymptotics for Λ_ρ and H_C in the long wave region (Eqs. (4.35) and (4.36) coincide with Eqs. (5.3) and (5.4) for small k_0a).

After substituting Eqs. (5.3) and (5.4) into the dispersion equation (3.16) and (3.17) and taking into account only the main terms in real and imaginary parts of its solution we obtain

$$k_* = k_s - i\gamma, \qquad k_s = \omega\sqrt{\frac{\rho_s}{\mu_s}}, \qquad \gamma = \frac{p\pi}{8}(k_sa)^3\left[\frac{\rho_1(\rho-\rho_s)}{\rho_s^2} + \frac{2\mu_1(\mu-\mu_s)}{(\mu+\mu_s)^2 - 2p\mu_1\mu}\right]. \tag{5.5}$$

Here μ_s and ρ_s are the "static" values of the effective shear modulus and density when ω, $k_0 = 0$. These parameters take the following forms

$$\rho_s = \rho_0 + p(\rho - \rho_0), \qquad \mu_s = \mu_0 + 2p\frac{(\mu-\mu_0)\mu_s}{\mu+\mu_s}. \tag{5.6}$$

The last equation is in fact an algebraic equation for the static shear modulus μ_s. This equation for the effective shear moduli of fiber reinforced composites was firstly obtained in Hill (1965a). For the absolutely rigid inclusions ($\mu \to \infty$) the solution of Eq. (5.6) is

$$\mu_s = \frac{\mu_0}{1-2p}, \tag{5.7}$$

and for cylindrical pores ($\mu = 0$)

$$\mu_s = (1-2p)\mu_0. \tag{5.8}$$

Note that these equations give physically reasonable values of μ_s ($\mu_s > 0$) only for $p < 0.5$.

Let us consider the solution of the dispersion equations (3.16) and (3.17) of the EMM in the short wave limit. In this case $\omega, k_0 \to \infty$, and as it follows from Eqs. (4.8), (4.25) and (4.26) $\Lambda_\rho, H_C \to 0$. As a result, the solution of the dispersion equations (3.16) and (3.17) for k_* takes the following form:

$$k_* = \omega\sqrt{\frac{\rho_0 + p\rho_1\Lambda_\rho}{\mu_0 + p\mu_1 H_C}} \approx k_0\left[1 - \frac{p}{2}\left(\frac{\mu_1}{\mu_0}H_C - \frac{\rho_1}{\rho_0}\Lambda_\rho\right)\right]. \tag{5.9}$$

Hence, when $k_0 \to \infty$ the attenuation factor γ is

$$\gamma = -\mathrm{Im}k_* = \frac{p}{2}k_0\mathrm{Im}\left(\frac{\mu_1}{\mu_0}H_C - \frac{\rho_1}{\rho_0}\Lambda_\rho\right) = \frac{pQ(k_0)}{2\pi a^2}, \tag{5.10}$$

where $Q(k_0)$ is the total scattering cross-section of the inclusion defined in Eq. (4.23). Taking into account Eqs. (4.25) and (4.26) we obtain for the short wave limits of the attenuation factor γ and phase velocity v_* of the mean wave field the following equations:

$$\lim_{k_0\to\infty} \gamma = \overline{\gamma} = \frac{2p}{\pi a}, \tag{5.11}$$

$$\lim_{k_0\to\infty} v_* = \lim_{k_0\to\infty} \frac{\omega}{\mathrm{Re}\, k_*} = \sqrt{\frac{\mu_0}{\rho_0}} = v_0. \tag{5.12}$$

Here we take into account that

$$k_0 \mathrm{Re}\left(\frac{\mu_1}{\mu_0} H_C - \frac{\rho_1}{\rho_0} \Lambda_\rho\right) \to 0 \quad \text{if } k_0 \to \infty. \tag{5.13}$$

Thus, the short wave limit of the velocity of the mean wave field coincides with the wave velocity in the matrix v_0. The short-wave limit of the attenuation factor γ does not depend on elastic properties of the matrix and inclusions and is proportional to the volume concentration of inclusions p. This result may be interpreted as follows. In the short wave limit the geometrical optic interpretation may be used for the description of the mean wave field in the composite. The mean field may be considered as a set of independent beams propagating through the medium. Because of existing a continuous component (matrix) the phase velocity of the mean wave field should coincide with the wave velocity in the matrix. The attenuation factor γ in the short wave limit does not depend on the frequency of the incident field and on the properties of the inclusions and is only a function of a number of scatterers in a unit length (see similar results for elastic, scalar and electromagnetic waves in Bussemer et al. (1991) and Kanaun (1996, 1997, 2000)).

Note that for many materials attenuation at high frequencies is mainly defined by viscosity and non-linear mechanisms that are not taken into account in this study.

6. The second version of the EMM

Version I of the EMM was applied to the calculation of the elastic moduli of composites with spherical inclusions in Budiansky (1965) and Hill (1965b). It turns out that the corresponding formulas for the elastic moduli of the composites with spherical inclusions do not correspond to experimental data in the region of high volume concentrations of inclusions ($p > 0.3$) (see, e.g., Kanaun and Levin, 1994). In order to correct the predictions of the EMM in this region another version of the EMM was proposed in Christensen and Lo (1979). A similar version of the EMM was considered in Kerner (1956) also for the case of statics. In this version (version II) the layer of the matrix material was involved in the border between the inclusion and the effective medium. Thus, the main hypothesis (1) of the method was formulated as follows

1^*. *Every inclusion in the composite behaves as a kernel of a layered inclusion embedded in the effective medium. The size and the properties of the kernel coincide with these characteristics of the inclusion, and the properties of the outside layer coincide with the properties of the matrix.*

The size of the outside layer depends on the volume concentration p of the inclusions. For cylindrical inclusions with circular cross-sections the radius of the kernel a and outside radius b of the matrix layer are connected by the equation (Kerner, 1956; Christensen and Lo, 1979)

$$\left(\frac{a}{b}\right)^2 = p. \tag{6.1}$$

The condition of self-consistency in this version coincides with hypothesis 2 of version I of the EMM.

In this version the one particle problem is the diffraction of a plane monochromatic wave on a layered inclusion embedded in a homogeneous effective medium. The differential equations of this problem are similar to Eq. (4.1) (for the field inside the inclusion and in the effective medium). The field in the matrix layer satisfies the equation $\Delta u + k_0^2 u = 0$, $a > |y| > b$, with the conditions similar to (4.2) on the boundaries of the layer. The same technique as in the case of a homogeneous inclusion gives us the following equations for the wave field $u(y)$ in the inclusion, the layer, and the medium

$$u(y) = u^+(y) = \sum_{m=0}^{\infty} a_m J_m(kr)\cos(m\varphi), \quad 0 \leqslant r \leqslant a, \tag{6.2}$$

$$u(y) = \sum_{m=0}^{\infty} [c_m J_m(k_0 r) + d_m N_m(k_0 r)]\cos(m\varphi), \quad a \leqslant r \leqslant b, \tag{6.3}$$

$$u(y) = u^-(y) = \sum_{m=0}^{\infty} [\epsilon_m (-\mathrm{i})^m U^* J_m(k_* r) + b_m H_m(k_* r)]\cos(m\varphi), \quad r > b. \tag{6.4}$$

Here $N_m(z)$ is the Bessel function of the second kind and of the order m. The constants a_m, b_m, c_m and d_m are to be found from the boundary conditions on the interfaces $r = a$ and $r = b$ that are similar to Eq. (4.2). These conditions give us a system of linear algebraic equations for the constants in Eqs. (6.2)–(6.4), which solution takes the form

$$a_m = \frac{1}{\Delta}(B_1 A_{22} - B_2 A_{12}), \qquad b_m = -\frac{1}{\Delta}(B_1 A_{21} - B_2 A_{11}), \tag{6.5}$$

$$c_m = \frac{\pi}{2\mu_0} A_{11} a_m, \qquad d_m = -\frac{\pi}{2\mu_0} A_{21} a_m, \tag{6.6}$$

$$\Delta = A_{11}A_{22} - A_{12}A_{21}, \tag{6.7}$$

$$A_{11} = \mu_0 k_0 a J_m(ka) N'_m(k_0 a) - \mu k a J'_m(ka) N_m(k_0 a), \tag{6.8}$$

$$A_{12} = \mu_* k_* b N_m(k_0 b) H'_m(k_* b) - \mu_0 k_0 b H_m(k_* b) N'_m(k_0 b), \tag{6.9}$$

$$A_{21} = \mu_0 k_0 a J_m(ka) J'_m(k_0 a) - \mu k a J'_m(ka) J_m(k_0 a), \tag{6.10}$$

$$A_{22} = \mu_* k_* b J_m(k_0 b) H'_m(k_* b) - \mu_0 k_0 b H_m(k_* b) J'_m(k_0 b), \tag{6.11}$$

$$B_1 = \epsilon_m (-\mathrm{i})^m U_* [\mu_0 k_0 b J_m(k_* b) N'_m(k_0 b) - \mu_* k_* b J'_m(k_* b) N_m(k_0 b)], \tag{6.12}$$

$$B_2 = \epsilon_m (-\mathrm{i})^m U_* [\mu_0 k_0 b J_m(k_* b) J'_m(k_0 b) - \mu_* k_* b J'_m(k_* b) J_m(k_0 b)]. \tag{6.13}$$

The dispersion equation of version II of the EMM is similar to the dispersion equation of version I and has the form of Eqs. (3.16) and (3.17). The functions Λ_ρ and H_C in (3.16) and (3.17) are given by Eqs. (4.8)–(4.10), where the coefficients a_m should be taken from Eq. (6.5).

Let us consider the solution of the dispersion equation of this version of EMM in the long wave region when $k_0 a \ll 1$. Using Eq. (5.1) one can find the main terms of the coefficients a_m in Eq. (6.5) in the forms

$$a_0 = 1 + \frac{\mathrm{i}\pi}{4}(k_* b)^2 \left\{ 1 - \frac{v_*^2}{\mu_*}\left[(1-p)\frac{\mu_0}{v_0^2} + p\frac{\mu}{v^2} \right] \right\},$$

$$a_1 = -\frac{8ik_*}{k}\mu_0\mu_*\left[a_{10} + \frac{i\pi}{4}(k_*b)^2 a_{11}\right], \tag{6.14}$$

$$a_{10} = \{\mu[(1+p)\mu_0 + (1-p)\mu_*] + \mu_0[(1-p)\mu_0 + (1+p)\mu_*]\}^{-1},$$

$$a_{11} = a_{10}^2\{\mu[(1+p)\mu_0 - (1-p)\mu_*] + \mu_0[(1-p)\mu_0 - (1+p)\mu_*]\}. \tag{6.15}$$

As a result, the functions Λ_ρ and H_C in Eqs. (3.16) and (3.17) for version II of the EMM take the forms

$$\Lambda_\rho = 1 - \frac{i\pi}{4}(k_*b)^2\frac{(1-p)\rho_0 + p\rho - \rho_*}{\rho_*}, \tag{6.16}$$

$$H_C = 4\mu_0\mu_*\left[a_{10} + \frac{i\pi}{4}(k_*b)^2 a_{11}\right]. \tag{6.17}$$

In the long wave region the main terms of the effective shear modulus μ_* and the effective density ρ_* should be found in the forms

$$\mu_* = \mu_s + \frac{i\pi}{4}(k_sb)^2\mu_\omega, \qquad \rho_* = \rho_s + \frac{i\pi}{4}(k_sb)^2\rho_\omega, \qquad k_s = \omega\sqrt{\frac{\rho_s}{\mu_s}}, \tag{6.18}$$

where μ_s, ρ_s are "static" values of these parameters ($\omega = 0$). The main term in the real and imaginary parts of functions Λ_ρ and H_C in Eqs. (6.16) and (6.17) are

$$\Lambda_\rho = 1 - \frac{i\pi}{4}(k_sb)^2\frac{(1-p)\rho_0 + p\rho - \rho_s}{\rho_s}$$

$$H_C = \frac{4\mu_0\mu_s}{\Delta_s}\left[1 + \frac{i\pi}{4}(k_sb)^2\left(\frac{\mu_0 d_1}{\mu_s}\mu_\omega + \mu_0 d_1 - \mu_s d_2\right)\right] \tag{6.19}$$

$$d_1 = (1+p)\mu + (1-p)\mu_0, \qquad d_2 = (1-p)\mu + (1+p)\mu_0, \qquad \Delta_s = \mu_0 d_1 + \mu_s d_2. \tag{6.20}$$

After substituting Eq. (6.19) into Eqs. (3.16) and (3.17) we obtain

$$\rho_s = p\rho + (1-p)\rho_0, \qquad \rho_\omega = p\frac{\rho_1}{\rho_s}[p\rho + (1-p)\rho_0 - \rho_s] = 0. \tag{6.21}$$

Thus, the imaginary part of the dynamic density ρ_ω in Eq. (6.18) turns to be equal to zero. It means that the series of the dynamic density with respect to frequency ω does not contain terms proportional to ω^2.

From Eqs. (3.16), (3.17) and (6.19) we also obtain that the static shear modulus μ_s satisfies the equation

$$\mu_s = \mu_0\left(1 + \frac{4p\mu_1\mu_s}{\Delta_s}\right) \tag{6.22}$$

and the imaginary part of the modulus μ_* in Eq. (6.18) takes the form

$$\mathrm{Im}(\mu_*) = \frac{\pi}{4}(k_sb)^2\mu_\omega, \qquad \mu_\omega = (\mu_0 d_1 - \mu_s d_2)\left[1 - 4p\mu_1 d_1\left(\frac{\mu_0}{\Delta_s}\right)^2\right]^{-1}. \tag{6.23}$$

Here coefficients d_i and Δ_s are defined in Eq. (6.20). Eq. (6.22) is a square algebraic equation for the modulus μ_s. It is easy to show that the only positive root of this equation is

$$\mu_s = \mu_0\left[1 + \frac{2p(\mu - \mu_0)}{2\mu_0 + (1-p)(\mu - \mu_0)}\right]. \tag{6.24}$$

Eq. (6.24) coincides with one of the well known Hashin–Strikman bounds for the effective shear modulus of the fiber composites (see Talbot and Willis, 1983). This equation also coincides with the result of application of another self-consistent method (the effective field method) to the problem under consideration (see Kanaun and Levin, 1993). Eq. (6.24) gives physically reasonable values of the effective shear modulus ($\mu_s > 0$) for all possible values of the volume concentrations of fibers and elastic properties of the latter.

Theoretical dependences of the shear modulus μ_s of a composite reinforced with cylindrical fibers on volume concentrations of the fibers p are compared with experimental data in Fig. 2. The composite with $\mu_0 = 2.03$ GPa, $\mu = 12.5$ GPa and $0 < p < 0.8$ is considered. The solid line in Fig. 2 is the prediction of version I of the EMM (Eq. (5.6)); the line with circles is the predictions of version II (Eq. (6.24)), squares are experimental data given in Dean and Lockett (1973). It is seen that the predictions of version II are closer to the experimental data than the predictions of version I in the region of high volume concentrations of inclusions.

After substituting Eq. (6.24) into the right-hand side of Eq. (6.23) we obtain that the imaginary part of the dynamic shear modulus μ_* disappears ($\mu_\omega = 0$). It follows from this equation and Eq. (6.21) ($\rho_\omega = 0$) that the series of the imaginary part of the effective wave number k_* (or the attenuation factor) with respect to ω begins with the terms of the order higher than ω^3 ($\gamma = o(\omega^3)$). Thus, version II of the EMM does not describe the attenuation caused by the Rayleigh scattering of waves on inclusions that is proportional to ω^3. This conclusion is independent of the volume concentration p of inclusions and their properties.

Note that in the literature exists another version of the EMM (version III) (see, e.g., Stroud and Pan, 1978). In this version hypothesis 1 coincides with hypothesis 1^* of version II but the condition of self-consistency (hypothesis 2) is formulated as following.

2^*. *The parameters of the effective medium are to be chosen in order to eliminate the forward amplitude of the wave field scattered on the layered inclusion embedded in the effective medium.*

As it follows from Eqs. (4.13), (4.14) and (6.4) the forward amplitude $A(\mathbf{n}^0)$ of the field scattered on the layered inclusion takes the form

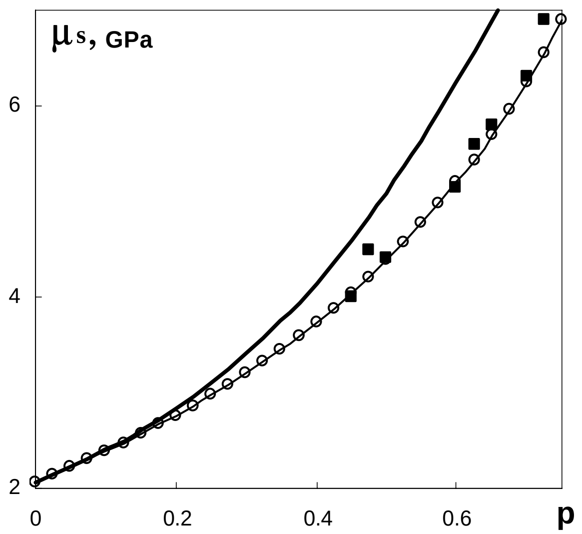

Fig. 2. The dependences of the static elastic shear modulus $\mu_* = \mu_s$ of the fiber composites on volume concentrations of inclusions p; the solid line corresponds to version I the EMM, the line with dots to version II, squares are experimental data in Dean and Lockett (1973).

$$A(\mathbf{n}^0) = \sum_{m=0}^{\infty} b_m \exp\left[\frac{i\pi}{2}(m+1)\right]. \tag{6.25}$$

According to hypothesis 2* the properties of the effective medium should be taken in order to eliminate the right-hand side of this equation. It is not difficult to realize this version of the EMM using Eq. (6.25). It turns out that the predictions of this version are very close to the predictions of the version II of the EMM developed in this study.

Let us show that version III also does not describe attenuation caused by Rayleigh wave scattering on inclusions. In the long wave region the main term of the right-hand side of Eq. (6.25) takes the form

$$A(\mathbf{n}^0) = b_0 - 2b_1 + i\pi(bk_s)^2(b_0^2 - 2b_1^2), \tag{6.26}$$

$$b_0 = \frac{1}{4\rho_*}[\rho_* - p\rho - (1-p)\rho_0], \qquad b_1 = \frac{\mu_0 d_1 - \mu_* d_2}{\mu_0 d_1 + \mu_* d_2}.$$

Here coefficients d_1 and d_2 are defined in Eq. (6.20). If $\rho_* = \rho_s,\ \mu_* = \mu_s$, where ρ_s and μ_s are given in Eqs. (6.21) and (6.24), the coefficients b_0, b_1 in Eq. (6.26) turn to be equal to zero. It means that the roots of the equation $A(\mathbf{n}^0) = 0$ in the long wave region are the static density and elastic modulus of the composite given in Eqs. (6.21) and (6.24). In the other words, the main terms of the dynamic density and shear modulus in the long wave region do not contain terms proportional to ω^2. Thus, version III of the EMM, similar to version II, predicts that the main term of the imaginary part of the effective wave number (attenuation factor) has the order higher than ω^3.

7. Numerical solution of the dispersion equation

In this section, we construct numerical solutions of the dispersion equations (3.16), and (3.17) of the EMM in the region $0 < k_0 a < 100$ of the wave numbers of the incident field. In the calculations we take $\mu_0 = 1$, $\rho_0 = 1$, $a = 1$ and for these data parameter $k_0 a$ coincides with the frequency ω of the incident field. The numerical solution is obtained by the iterative procedure based on the equations that follow from Eqs. (3.16) and (3.17)

$$\begin{aligned} \mu_*^{(n)} &= \mu_*^{(n-1)} + \varepsilon[\mu_*^{(n-1)} - \mu_0(1 + p\overline{\mu}_1 H_C(k_*^{(n-1)}, \mu_*^{(n-1)}))], \\ \rho_*^{(n)} &= \rho_*^{(n-1)} + \varepsilon\left[\rho_*^{(n-1)} - \rho_0[1 + p\overline{\rho}_1 \Lambda_\rho(k_*^{(n-1)}, \mu_*^{(n-1)})]\right], \\ k_*^{(n)} &= \omega\left(\frac{\rho_*^{(n)}}{\mu_*^{(n)}}\right)^{1/2}, \qquad \overline{\mu}_1 = \frac{\mu_1}{\mu_0}, \qquad \overline{\rho}_1 = \frac{\rho_1}{\rho_0}. \end{aligned} \tag{7.1}$$

Here $k_*^{(n)}$, $\mu_*^{(n)}$, $\rho_*^{(n)}$ are the effective parameters for the nth iteration; functions $H_C(k_*, \mu_*)$ and $\Lambda_\rho(k_*, \mu_*)$ are defined in Eqs. (4.8)–(4.10). Parameter $\varepsilon(|\varepsilon| < 1)$ is to be chosen for conversion of the iterative process. For version I of the EMM the coefficient a_m in Eqs. (4.8)–(4.10) are defined in Eqs. (4.6) and (4.7), and for version II a_m are given in Eqs. (6.5)–(6.13). As an initial (zero) approximation we use the static solution for μ_* ($\mu_*^{(0)} = \mu_s$) given in Eq. (5.6) and the equation $k_*^{(0)} = \omega\sqrt{(\rho_0 + p\rho_1)/\mu_s}$ for the effective wave number.

The dependences of the velocities and attenuation factors of the mean wave field on the frequency of the incident field and volume concentrations of inclusions are presented in Figs. 3 and 4(a)–(c). In these figures solid lines correspond to version I of the EMM, the lines with circles to version II, and the lines with triangles correspond to version I when the approximate solution of the one particle problem (see Section 4.2) is used. The cases of hard and heavy inclusions ($\mu/\mu_0 = 100$, $\rho/\rho_0 = 10$) are in Fig. 3(a)–(c) (for three values of the volume concentrations of fibers $p = 0.1; 0.3; 0.5$), and the cases of soft and light inclusions ($\mu/\mu_0 = 0.01$, $\rho/\rho_0 = 0.1$) are in Fig. 4(a)–(c). Horizontal dashed lines in Figs. 3 and 4(a)–(c) are the short

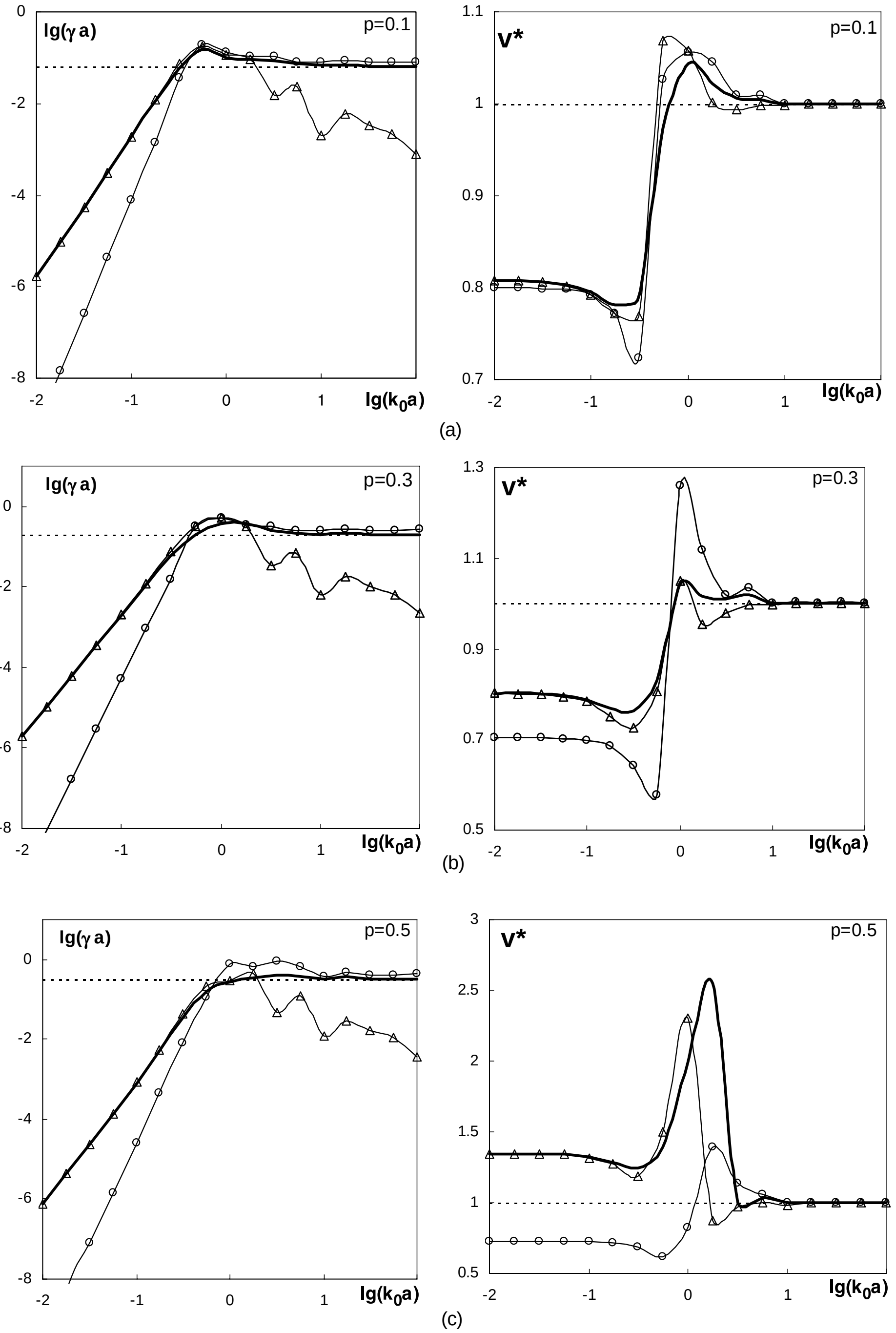

Fig. 3. (a) The dependences of attenuation factors γ and velocities of the mean wave field v in the composites with hard and heavy inclusions ($\mu_0 = 1$, $\mu = 100$, $\rho_0 = 1$, $\rho = 10$) on the frequency of the incident field, volume concentration of fibers $p = 0.1$. Solid lines corresponds to the first version of the EMM, lines with dots to the second version of the EFM, lines with triangles correspond the first version and approximate solution of the one particle problem; (b) the same graphs as in Fig. 3a for $p = 0.3$; (c) the same graphs as in Fig. 3a for $p = 0.5$.

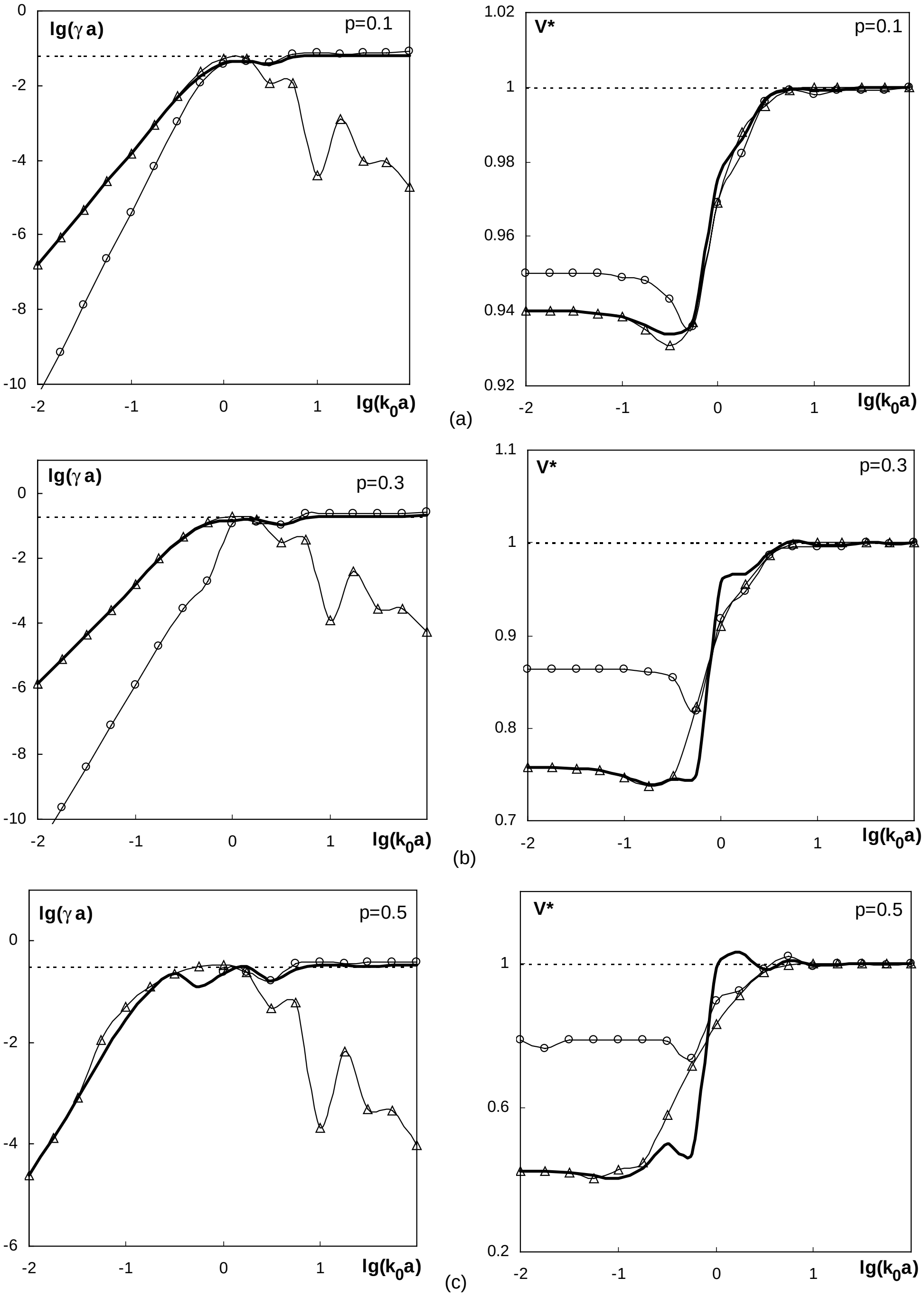

Fig. 4. (a) The dependences of attenuation factors γ and velocities of the mean wave field v in the composites with soft and light inclusions ($\mu_0 = 1$, $\mu = 0.01$, $\rho_0 = 1$, $\rho = 0.1$) on the frequency of the incident field, volume concentration of fibers $p = 0.1$. Solid lines corresponds to the first version of the EMM, lines with dots to the second version of the EFM, lines with triangles correspond the first version and approximate solution of the one particle problem; (b) the same graphs as in Fig. 4a for $p = 0.3$; (c) the same graphs as in Fig. 4a for $p = 0.5$.

wave limits of the attenuation factors and of the mean wave velocities for the considered volume concentration of inclusions (see Eqs. (5.11), and (5.12)). Note that short wave limits of v_* and γ are practically achieved for $k_0a = 30$.

8. Discussion and conclusion

The analytical and numerical results obtained in previous sections allow us to compare the predictions of versions I and II of the EMM in a wide region of frequencies of the incident field, volume concentrations and properties of inclusions. In the cases of contrasted matrix and inclusions ($\mu/\mu_0, \rho/\rho_0 > 10$ or $\mu/\mu_0, \rho/\rho_0 < 0.1$) and for small volume concentrations of the latter ($p < 0.2$) both methods give close results for the velocities of the mean wave fields in the composites. But these versions predict quite different behavior of the attenuation factor γ of the mean wave field in the long wave region. In this region γ has the order of ω^3 for the first version of the EMM and the order of ω^5 for the second version. As a result, the corresponding dependences of γ on k_0 have different slopes in the logarithmic scale (see Figs. 3 and 4(a)–(c)). Thus, version II does not describe Rayleigh scattering of waves on inclusions that takes place in any homogeneous medium with a random set of isolated inclusions in the long wave region. In the middle and short wave regions both versions of the EMM give close results for the attenuation factors.

The algorithm of the EMM is simplified essentially if the approximate solution of the one particle problem is used (Section 4.2). The EMM with the approximate and exact solutions of the one particle problem give close predictions for the velocities and attenuation factors of the mean wave field in the long wave region ($k_0a < 1$), but these predictions deviate in the short wave region. The approximate solution is based on the assumption that the wave field inside every inclusion is constant. This assumption is evidently violated in the short and middle wave regions. As a result, in these regions the field inside an inclusion and the field scattered by an inclusion are calculated with essential error. In fact, an inclusion scatters more energy than the approximate solution predicts. This error increases with the frequency, and as a result, the attenuation factor decreases instead of being constant in the short wave region (see Figs. 3 and 4(a)–(c)). Nevertheless the approximate solution describes the dependences of the velocity of the mean wave field on frequency in all frequency regions sufficiently well.

Essential discrepancies of two versions of the EMM may be observed for high volume concentrations of inclusions ($p > 0.3$) in the low and middle wave region ($k_0a < 5$). In the long wave region the velocity of the mean wave field is mainly defined by static elastic properties of the composite. As it was shown in Section 6, version II of EMM corresponds better to experimental data in case of statics. Thus, one can expect that the predictions of this version for the velocities of the mean wave field are more reliable than the predictions of version I in the long wave region if $p > 0.3$.

The most abrupt changes in the dependences of the velocity of the mean wave field on frequency are observed in the region where $k_0a = O(1)$. This is the region of the first quasi-resonance of an isolated inclusion in homogeneous matrix (see the first maxima in the frequency dependences of the total scattering cross-section of isolated inclusions in Fig. 1). After that, in the middle wave region, the structure of the mean wave field becomes more complex, and one observes more oscillations in the frequency dependences of the velocity in this area. But the changes near the first quasi-resonance are stronger than in any other region. Experiments show a similar behavior of the frequency dependences of the velocity of the mean wave field in the composites with spherical inclusions (see Sabina and Willis, 1988).

If the inclusions and matrix have less contrast $0.1 < \mu/\mu_0$, $\rho/\rho_0 < 10$, the predictions of both versions of the EMM for the velocities of the mean wave field become closer, and the process of numerical solution of the dispersion equation (see Section 7) needs less iterations. Nevertheless the predictions of the two version of the EMM for attenuation factors in the long wave region are different even for non-contrasted matrix and inclusions.

As a final conclusion one can make the following points. The EMM is an efficient way to calculate the velocities and attenuation factors of the mean wave fields in fiber reinforced composite materials. The algorithm of calculation of these parameters is fast, simple, and does not create difficulties in programming. The predictions of version I of the EMM are reliable only in the region of small volume concentrations of inclusions ($p < 0.2$). The error of the calculation of the velocities as well as the attenuation factors increases with the volume concentration of inclusions if this version is used. Version II of the EMM improves the predictions of the velocities of the mean wave field in the region of high volume concentrations of inclusions but attenuation factors in the long wave region are calculated with essential errors.

It is necessary to note that an inevitable defect of all the versions of the EMM is their inability to describe the influence of the peculiarities in spatial distributions of inclusions in the matrix on the effective properties of the composite material. Such a description is possible in the framework of another self-consistent scheme: the so-called effective field method (see Kanaun and Levin, 1993, 1994; Kanaun, 2000).

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